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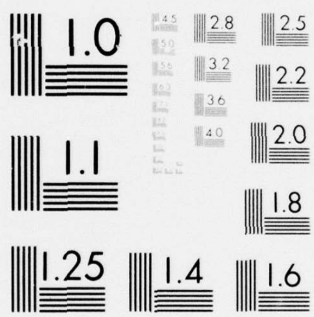

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THE M/G/1 QUEUE WITH INSTANTANEOUS, BERNOULLI FEEDBACK

Ralph L. Disney

and

Donald C. McNickle

Corrigendum

The formula at the bottom of page 7 and the parenthetical term two lines above are in error. The T'_n there should be replaced by $T'_n - T'_{n-1}$.

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by

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and

Donald C. McNickle

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In this paper we are concerned with several random processes that occur in M/G/1 queues with instantaneous feedback in which the feedback decision process is a Bernoulli process. In particular, we study the queue length processes: a) at arbitrary times, $Q(t)$; b) at departure points, $Q_4^+(n)$; c) at output times, $Q_3^+(n)$; d) at arrival times, $Q_1^-(n)$; e) at input times, $Q_2^-(n)$. It is shown that $Q(t)$, $Q_1^-(n)$, $Q_4^+(n)$ are asymptotically identically distributed as are $Q_3^+(n)$ and $Q_2^-(n)$. However, $Q(t)$ and $Q_3^+(n)$ are not so distributed.

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Departure, output, input and feedback random processes are also studied. It is shown that except for the departure process in M/M/1 queues none of these are renewal processes as long as there is some non-zero probability of feedback occurring. Implications of these results for network decomposition are discussed.

THE M/G/1 QUEUE WITH INSTANTANEOUS,
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is

1.0 Introduction. In this paper we are concerned with several random processes that occur within the class of M/G/1 queues with instantaneous feedback in which the feedback decision process is a Bernoulli process. Such systems in the case G=M are the simplest, non-trivial examples of Jackson networks [5]. Indeed, they are so simple that they are usually dismissed from consideration in queueing network theory as being obvious. We will show that far from being obvious they exhibit some important, unexpected properties whose implications raise some interesting questions about Jackson networks and their application.

It will be shown

In particular, Jackson [5] observed that in his networks the vector valued queue length process behaved as if the component processes were independent, M/M/1 systems. Since those results appeared there has developed a mythology to explain them. These arguments usually rest on three sets of results that are well known in random point process theory: superposition, thinning, and stretching. By examining the network flow, it will be shown that the application of these results are inappropriate to queueing network with instantaneous, Bernoulli feedback. Their flows are considerably more complicated than one expects based on such arguments and one is left to ponder what the Jackson results mean to queueing network decomposition.

1.1 The Problem and Notation. We assume the usual apparatus of M/G/1 queues with unlimited waiting capacity. The new idea is that a unit which has received service departs with probability q and returns for more service with probability p. p+q=1. Without loss of generality for the processes studied here the returning customer can be put anywhere in the queue.

To establish notation it is assumed that the arrival process is a Poisson process with parameter $\lambda>0$. The arrival epochs are the elements of $\{T_n: n=1,2,\ldots\}$. Service times are i.i.d. random variables, S_n with

$$\Pr[S_n \le t] = H(t),\ t \ge 0,$$

$$E[S_n] < \infty.$$

The arrival process and service times are independent processes.

Service completions occur at $T_0<T_1<T_2\ldots$ called the <u>output epochs</u>. Let

$$Y_n = Y_{T_n} = \begin{cases} 0, \text{ if the n-th output departs} \\ 1, \text{ if the n-th output feeds back.} \end{cases}$$

$\{Y_n\}$ is a Bernoulli process.

Elements of the subset $\{t_n\}\subset\{T_n\}$ are called the <u>departure epochs</u> and are the times at which an output leaves the system. The elements of the subset $\{\tau_n\}\subset\{T_n\}$ are called the <u>feedback epochs</u> and are the times at which an output returns to the queue. $\{t_n\}\cup\{\tau_n\} = \{T_n\}$.

The times T'_n are the times at which a unit enters the queue. $\{T'_n\}$ is called the <u>input process</u>. $\{T'_n\} = \{T_n\}\cup\{\tau_n\}$.

There are six queue length processes to be studied. They are closely related as will be shown. Thus, let

$$Q(t) = \text{the queue length (number in the system) at } t.$$

Then, $Q_1^-(n)=Q(T_n-0)$; $Q_2^-(n)=Q(T'_n-0)$; $Q_3^+(n)=Q(T_n+0)$; $Q_4^+(n)=Q(t_n+0)$ are respectively the embedded queue lengths at arrival epochs, input epochs, output epochs, departure epochs.

2.0 Queue Length Processes. The queue lengths listed in 1.1 are closely related. The steady state versions of $\{Q_3^+(n)\}$ and $\{Q_4^+(n)\}$ are of primary concern. They are studied in sections 2.1 and 2.2 separately. They are related to the other processes in section 2.3. The important special case for G=M is then studied in 2.4.

2.1 The $\{Q_4^+(n)\}$ Process. There are several ways to study this process. The following appears to be direct, correct and may help explain why these feedback problems have received such little attention in the queueing literature. First it is clear that

$$t_n = \begin{cases} t_{n-1} + S_n', \text{ if } Q_4^+(n-1)>0, \\ t_{n-1} + I_n + S_n', \text{ if } Q_4^+(n-1)=0. \end{cases}$$

Here S_n' is the total service time consumed between the (n-1)-st and n-th departure. I_n is the idle time following t_{n-1}. For the M/G/1 queues $\{I_n\}$ are i.i.d. random variables that are exponentially distributed with parameter λ.

Without loss of generality, since customers are indistinguisable,

$$S_n' = S_1 + S_2 + \dots S_m, \text{ for every n.}$$

m is the number of services performed between the n-th and (n-1)-st departure. Since $\{Y_n\}$ is a Bernoulli process, m is geometrically distributed and it follows that $\{S_n'\}$ is a sequence of i.i.d. random variables. Thus the Laplace transform of the distribution function of S_n' is easily found to be

$$G^*(s) = qH^*(s)/[1-pH^*(s)].$$

Using standard embedded Markov chain methods one finds that the probability generating function of Π' the limiting probability distributions of $\{Q_4^+(n)\}$, is given by

$$g'(z) = \frac{\Pi'(0)(z-1)G^*(\lambda-\lambda z)}{z-G^*(\lambda-\lambda z)} \tag{2.1.1}$$

and

$$\Pi'(0) = 1 - \lambda E[S_n]/q. \tag{2.1.2}$$

Since (2.1.1) and (2.1.2) are of the form one finds for M/G/1 queues without feedback, by properly adjusting the mean service time, it can be argued that the queue length process embedded at departure points for queues with and without feedback are asymptotically identically distributed. In fact, this result is true for all n.

2.2. The $\{Q_3^+(n)\}$ Process. This is the queue length process embedded at output points. Since $\{t_n\} \subset \{T_n\}$ $\{Q_4^+(n)\}$ is a process on a courser grid than $\{Q_3^+(n)\}$. Since one is ultimately to be concerned with both $\{Q_3^+(n)\}$ and $\{T_n - T_{n-1}\}$, the following study is for the joint process $\{Q_3^+(n),\ T_n - T_{n-1}\}$. The marginal results for $\{Q_3^+(n)\}$ then will be easy to determine.

Theorem 2.2.1. The process $\{Q_3^+(n), T_n - T_{n-1}\}$ is a Markov Renewal process with transition functions $A(i,j,x) = \Pr[Q_3^+(n)=j,\ T_n - T_{n-1} \le x \mid Q_3^+(n-1)=i]$ where

$$A(i,j,x) = \begin{cases} 0\,, & j < i-1, \\ \int_0^x qe^{-\lambda y}\,dH(y), & i \neq 0,\ j=i-1, \\ \int_0^x \frac{(\lambda y)^{j-i}}{(j-i)!}\, e^{-\lambda y}\,[p + \frac{q\lambda y}{(j-i+1)}]\, dH(y), & i\neq 0,\ j > 1, \\ \int_0^x [1-e^{-\lambda(x-y)}]\, \frac{(\lambda y)^{j-1}}{(j-1)!}\, e^{-\lambda y}\,[p + \frac{q\lambda y}{j}]\, dH(y), & i=0,\ j > 0, \\ \int_0^x [1-e^{-\lambda(x-y)}]\, e^{-\lambda y}\, q\,dH(y), & i = j = 0. \end{cases}$$

<u>Proof</u>.

$$T_n - T_{n-1} = \begin{cases} S_n, & \text{if } Q_3^+(n-1) > 0 \\ I_n + S_n, & \text{if } Q_3^+(n-1) = 0 \end{cases}$$

where I_n is the exponentially distributed idle time preceeding S_n if $Q_3^+(n-1)=0$. The result then follows directly using arguments as in [4].

If $x\to\infty$, $A(i,j,x) \to A(i,j)$ the one step transition probability for the $\{Q_3^+(n)\}$ process. Then using standard embedded Markov chain results one can show that the probability generating function $g(z)$ for the limiting probabilities $\Pi(j)$ are given by

$$g(z) = \frac{\Pi(0)(z-1)[pzH^*(\lambda-\lambda z)+qH^*(\lambda-\lambda z)]}{z-pzH^*(\lambda-\lambda z)-qH^*(\lambda-\lambda z)}$$

and

$$\Pi(0) = q - \lambda E[S_n].$$

2.3 <u>Other Queue Length Processes</u>. The queue length, limiting probabilities for the three queueing processes $\{Q_0(t)\}$, $\{Q_1^-(n)\}$, $\{Q_2^-(n)\}$ now follow from a theorem found in Cooper [3]. From this it follows that $\{Q_0(t)\}$; $\{Q_1^-(n)\}$; $\{Q_4^+(n)\}$ are asymptotically identically distributed and $\{Q_2^-(n)\}$; $\{Q_3^+(n)\}$ are asymptotically identically distributed. Clearly, $\{Q_4^+(n)\}$ and $\{Q_3^+(n)\}$ are not asymptotically, identically distributed.

The difference can be explained as follows. Any output must be either a feedback or a departure, if the queue length is j after an output then either there has been a feedback and the queue length is now j, or there has been a departure and the queue length is now j.

Thus $\Pi(j)=q\Pi'(j)+p(\Pr(\text{queue length after a feedback is } j))$. Since a queue length of j after a feedback, and a queue length of j-1 after a departure, both correspond to a queue length of j before the output, and so have the same probability

$$\Pi(j) = q\Pi'(j) + p\Pi'(j-1) \quad j \geq 1,$$

$$\Pi(0) = q\Pi'(0).$$

From the above argument and Cooper's theorem it can be seen that the probability of j in the queue just after an input (i.e., either a feedback or and arrival), is given by $\Pi'(j-1)$.

For the M/M/1 feedback queue, where

$$\Pi'(j-1) = \left(1-\frac{\lambda}{\mu q}\right)\left(\frac{\lambda}{\mu q}\right)^{j-1}, \quad j = 1,2,\ldots,$$

this result was pointed out in Burke [2].

2.4 The M/M/1 Case. If one assumes that the service time distribution is

$$H(t) = 1 - e^{-\mu t}, \quad t \geq 0,$$

some further classification is possible here. From the results of Jackson [5],

$$\Pi'(j) = (1 - \frac{\lambda}{q\mu})(\frac{\lambda}{q\mu})^j \quad , \; j = 0,1,2,\ldots \; .$$

From (2.1.1) and (2.1.2) one obtains

$$\Pi(0) = q(1 - \frac{\lambda}{q\mu}) \quad ,$$

$$\Pi(j) = (1 - \frac{\lambda}{q\mu})(\frac{\lambda}{q\mu})^{j-1} \; (p + \frac{\lambda}{\mu}), \; j = 1,2,\ldots$$

3.0 Flow Processes. To further clarify the problems here, it is useful to study the flow processes in this system. There are five processes of interest: the arrival process; the input process; the output process; the departure process; and the feedback process.

There has been some questions since the publication of the Jackson results concerning the interpretation of his results [1], [2]. In his paper Jackson showed that for his networks the joint limiting probability for the vector or queue lengths at each server could be factored into limiting probabilities for the queue length at each server. This implies that the queue lengths are independent in the limit. The remarkable result was that the marginal limiting probabilities were precisely those of an M/M/1 queue. Burke [2], has argued that the Jackson results are surprising. Burke's argument is based on showing that the input to a single server queue with feedback is not Poisson because the interinput times (our $\{T'_n\}$) are not exponentially distributed. [2] gives the precise result $(T'_n - T'_{n-1})$

$$\Pr[T'_n \leq x] = 1 - \frac{q\mu-\lambda}{\mu-\lambda} \; e^{-\lambda t} - \frac{p\mu e^{-\mu t}}{\mu-\lambda} \; , \; t \geq 0.$$

$(T'_n - T'_{n-1})$

In this section we will study some of the flows in this network and show indeed the Jackson results are surprising. A conjecture based on these results will be given in section 4.

3.1 Departures. The departure process $\{t_n\}$ can be studied as in Disney, Farrell, deMorais [4] upon using the mapping in section 2.1. Thus we know that this departure process is a renewal process and is a Poisson process whenever $\{S_n\}$ is a renewal process with exponential distribution. This is the Jackson case. So we conclude that the departure process from the Jackson network is a Poisson process.

From the results of section 2.1 there is a possibility that the departure process is Poisson even if S_n is not exponentially distributed. The result that is needed for the results of [4] to follow is that S_n' be exponentially distributed (since it is known that $\{S_n'\}$ is a sequence mutually independent, identically distributed random variables).

Theorem 3.1.1. If H(t) has a Laplace transform then the departure process from the M/G/1 queue with feedback is a Poisson process if and only if S_n is exponentially distributed for every n.

Proof. From section 2.1 we have $G^*(s)$, the Laplace transform of the distribution function H(t) is given by

$$G^*(s) = \frac{qH^*(s)}{1-pH^*(s)} \quad .$$

Thus if $G^*(s) = \frac{a}{a+s}$, S_n' is exponentially distributed with a parameter a and the departure process will be a Poisson process from [4]. Thus, if

$$\frac{a}{a+s} = \frac{qH^*(s)}{1-pH^*(s)}$$

S'_n is exponentially distributed. But the only solution here is

$$H^*(s) = \frac{a/q}{a/q+s}$$

which implies H(t) is exponential.

3.2 Outputs and Inputs. From section 2.2 it is clear that the output process is a Markov Renewal process whose distributions are given by A(i,j,x). From these, the following results are obtained.

Theorem 3.2.1. The output process $\{T_n - T_{n-1}\}$ is a renewal process if and only if $q = 1$ and $H(t) = 1 - e^{-\mu t}$.

Proof. If $q=1$ and $H(t) = 1-e^{-\mu t}$, the result follows from [4]. Those arguments can be modified in a trivial way to show that equations (3.1) and (3.2) of that paper are not satisfied jointly for any $q \neq 1$. Thus the output process is not renewal.

To be more specific, theorem 3.2.1 can be particularized as

Corollary 3.2.2. The output process $\{T_n - T_{n-1}\}$ for the M/M/1 queue is a Poisson process if and only if $q=1$.

Proof. Define

$$F(x) = Pr[T_n - T_{n-1} \leq x].$$

$F(x) = \Pi A U$ where U is the column vectors all of whose elements are 1, Π is the vector of limiting probabilities given in section 2.3 for $\{Q_3^+(n)\}$ and A is the matrix of A(i,j). Then from theorem 2.2.1 one obtains after some easy manipulations:

$$F(x) = (q - \frac{\lambda}{\mu}) \int_0^x [1-e^{-\lambda(x-y)}]\, dH(y) + (p + \frac{\lambda}{\mu})\, H(x) \tag{3.2.1}$$

for any M/G/1 queue with instantaneous, Bernoulli feedback.

For $H(y) = 1 - e^{-\mu y}$, it follows that

$$F(x) = 1 - \frac{q\mu-\lambda}{\mu-\lambda} e^{-\lambda x} - \frac{p\mu}{\mu-\lambda} e^{-\mu x}, \quad x \geq 0. \tag{3.2.2}$$

Thus single intervals are not exponentially distributed and the output process is not a Poisson process unless q=1.

Formula 3.2.2 was previously found by Burke [2] for the distribution of times between inputs. Whether this implies that the output random process and input random process are equivalent processes or not is not known. One conjectures that they are. The distribution form from (3.2.2) does yield

Corollary 3.2.3. The input process to the M/M/1 queue is not a renewal process unless q=1.

Proof. Assume that this input process is a renewal process with intervals distributed as in formula (3.2.2). From Cooper [3], the queue length process $\{Q_2^-(n)\}$ and $\{Q_3^+(n)\}$ should then be identically, asymptotically distributed as given in section 2.3. Using standard methods of studying GI/M/1 queues one finds that the unique root to the standard secular equation, which lies in (0,1) is given by

$$a = [(2\mu+\lambda) - [(2\mu-\lambda)^2 - 4p\mu^2]^{1/2}]/2\mu.$$

It then follows that $\{Q_3^+(n)\}$ and $\{Q_2^-(n)\}$ are asymptotically identically distributed if and only if q=1 (or p=0). Thus the input process is not a renewal process.

It seems obvious that the arrival process and feedback process are not independent processes. One can show:

<u>Corollary 3.2.4</u>. Either the feedback process is not a Poisson process or the arrival process and feedback process are not independent processes.

<u>Proof</u>. This result follows immediately from Burke's result on the distribution of the interinput intervals. For if the feedback process is both independent of the arrival process and is itself a Poisson process, the input process is Poisson. Thus Burke's result contradicts the assumption.

3.3 <u>Feedback</u>. The feedback stream seems to be quite difficult to work with. From the previous section we know that it is either not independent of the arrival stream or not a Poisson stream. We conjecture that both of these conditions prevail. If so then the known superposition theorems cannot be used to study feedbacks as part of the arrival, feedback, input processes. Treating the feedback stream as a filtered version of the Markov renewal output stream appears to be quite difficult. The only result for this feedback stream that we have is given by the following analysis.

Consider the probability that a sequence of n service times takes less than t, given that we start with i customers, $F_i^n(t)$. Let $H^n(t)$ be the distribution function of the sum of n service times. Either the first busy period terminates with one of these customers, or it does not. Thus

$$F_i^n(t) = \sum_{m=i}^{n-1} \int_0^t \frac{i}{m} e^{-\lambda z} \frac{(\lambda z)^{m-i}}{(m-i)!} \, dH^m(z) H^{n-m}(t-z)$$

+ Pr[n-th customer is served during the first busy period and before time t],

for $n > i$.

Otherwise, $F_i^n(t)$ = Pr[n-th customer served by time t]

Let $f_i(t)$ be the probability that a feedback occurs before time t, given the queue length just after a feedback was i.

Then

$$f_i(t) = \sum_{n=1}^{\infty} pq^{n-1} \int_0^t dH^n(z) - \sum_{n=i+1}^{\infty} \sum_{m=i}^{n-1} \int_0^t \frac{i}{m} e^{-\lambda z} \frac{(\lambda z)^{m-i}}{(m-i)!} dH^m(z)H^{n-m}(t-z)$$

$$+ \sum_{n=i}^{\infty} \int_{u=0}^{t} \int_{z=0}^{t-u} q^n \frac{i}{n} e^{-\lambda u} \frac{(\lambda u)^{n-i}}{(n-1)!} dH^n(u)\ \lambda e^{-\lambda(z)}\ f_i(t-u-z)$$

4.0 Conclusions. There are several conjectures that one can pose concerning Jackson networks based on the results of this paper. First with respect to queue length, busy period and departure processes if one adopts the "outsiders" view [3] these processes appear to be those generated by an M/G/1 queue without feedback. However, if one adopts the "insiders" view the queue length process does not appear to behave as seen by the "outsider".

Flow processes in this network cannot be explained by appeal to superposition, departure and random deletion results for Poisson processes. The requisite independence assumptions both within and between streams of events are not satisfied here. Thus one cannot assume that these queues which act "as if" they were M/M/1 queues to the "outsider" are M/M/1 queues to the "insider". In particular, this hints at the possibility that in these networks, even as simple as Jackson networks, any attempt to decompose the network into independent M/M/1 queues is doomed to failure. This decomposition must account for the internal flows and these not only appear to be non-Poisson, they appear to be non-renewal and are dependent on each other.

We conjecture, based on current papers in process, that if $p_{ii}^{(n)}>0$ for some i and n>0, in the Jackson structure, then flow along any path that returns a customer to a node that he has previously visited is not only not Poisson it is not renewal. Thus, if Jackson networks have loops, direct feedback as in this paper being the simplest example, they cannot be decomposed into sub-networks of simple M/M/1 servers.

In particular, these results probably imply that a node-by-node analysis of waiting times, depending as they do on the "insiders" view is not valid if one simply uses M/M/1 results at each server.

That these queues with feedback are far from trivial is clear. Considerably more research is needed to thoroughly understand what effect feedback has on queueing networks.

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